DISCRETE MODELS OF THE SELF-DUAL AND ANTI-SELF-DUAL EQUATIONS

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Abstract. In the case of a gauge-invariant discrete model of Yang-Mills theory difference self-dual and anti-self-dual equations are constructed.

1. Introduction

In 4-dimensional non-abelian gauge theory the self-dual and anti-self-dual connections are the most important extrema of the Yang-Mills action. Consider a trivial bundle $P = \mathbb{R}^4 \times G$, where G is some Lie group. We define a connection as some $\mathfrak{g}$-valued 1-form A, where $\mathfrak{g}$ is the Lie algebra of the group G [5]. Then the connection 1-form A can be written as follows

$$A = \sum_{a,\mu} A^a_\mu(x)\lambda_a dx^\mu, \tag{1}$$

where λ_a is the basis of the Lie algebra $\mathfrak{g}$. The curvature 2-form F of the connection A is given by

$$F = dA + A \wedge A. \tag{2}$$

We specialize straightaway to the choice $G = SU(2)$, then $\mathfrak{g} = su(2)$. We define the covariant exterior differentiation operator d_A by

$$d_A\Omega = d\Omega + A \wedge \Omega + (-1)^{r+1}\Omega \wedge A, \tag{3}$$

where Ω is an arbitrary $su(2)$-valued r-form. Compare (2) and (3) we obtain the Bianchi identity

$$d_AF = 0. \tag{4}$$

The Yang-Mills action S can be conveniently expressed (see [5, p. 256]) in terms of the 2-forms F and $*F$ as

$$S = -\int_{\mathbb{R}^4} tr(F \wedge *F),$$

Key words and phrases. Yang-Mills equations, discrete models, difference self-dual and anti-self-dual equations.

where $*$ is the adjoint operator (Hodge star operator). The Euler-Lagrange equations for the extrema of S are

$$d_A * F = 0. \tag{5}$$

Equations (4), (5) are called the Yang-Mills equations [4]. These equations are nonlinear coupled partial differential equations containing quadratic and cubic terms in A.

In more traditional form the Yang-Mills equations are expressed in terms of components of the connection A and the curvature F (see [2,3]). Let

$$A_\mu = \sum_\alpha A_\mu^\alpha(x)\lambda_\alpha$$

be the component of the connection 1-form (1). Then the components of the curvature form are given by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} + \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu],$$

where $[\cdot\ ,\cdot]$ be the commutator of the algebra Lie $su(2)$. In local coordinates the covariant derivative ∇_j can be written

$$\nabla_j F_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial x^j} + [A_j, F_{\mu\nu}].$$

Then we can write Equations (4), (5) as

$$\nabla_j F_{\mu\nu} + \nabla_\mu F_{\nu j} + \nabla_\nu F_{j\mu} = 0, \tag{6}$$

$$\sum_{\mu=1} \frac{\partial F_{\mu\nu}}{\partial x^\mu} + [A_\mu, F_{\mu\nu}] = 0. \tag{7}$$

Note that Equations 7 are obtained in the case of Euclidean space $\mathbb{R}^4$.

The self-dual and anti-self-dual connections are solutions of the following nonlinear first order differential equations

$$F = *F, \qquad F = - * F. \tag{8}$$

Equations (8) are called self-dual and anti-self-dual respectively. It is obviously that if one can find A such that $F = \pm * F$, then the Yang-Mills equations (5) are automatically satisfied.

2. The discrete model in $\mathbb{R}^4$

In [6] the gauge invariant discrete model of the Yang-Mills equations is constructed in the case of the n-dimensional Euclidean space $\mathbb{R}^n$. Following [6], we consider a combinatorial model of $\mathbb{R}^4$ as a certain 4-dimensional complex $C(4)$. Let $K(4)$ be a dual complex of $C(4)$. The complex $K(4)$ is a 4-dimensional complex of cochains with

$su(2)$-valued coefficients. We define the discrete analogs of the connection 1-form A and the curvature 2-form F as follows cochains

$$A = \sum_k \sum_{i=1}^{4} A_k^i e_i^k, \qquad F = \sum_k \sum_{i<j} \sum_{j=2}^{4} F_k^{ij} \varepsilon_{ij}^k, \tag{9}$$

where A_k^i, $F_k^{ij} \in su(2)$, e_i^k, ε_{ij}^k are 1-, 2-dimensional basis elements of $K(4)$ and $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$. We use the geometrical construction proposed by A. A. Dezin in [1] to define discrete analogs of the differential, the exterior multiplication and the Hodge star operator.

Let us introduce for convenient the shifts operator τ_i and σ_i as

$$\tau_i k = (k_1, ...\tau k_i, ...k_4), \qquad \sigma_i k = (k_1, ...\sigma k_i, ...k_4),$$

where $\tau k_i = k_i + 1$ and $\sigma k_i = k_i - 1$, $k_i \in \mathbb{Z}$. Similarly, we denote by τ_{ij} (σ_{ij}) the operator shifting to the right (to the left) two differ components of $k = (k_1, k_2, k_3, k_4)$. For example, $\tau_{12}k = (\tau k_1, \tau k_2, k_3, k_4)$, $\sigma_{14}k = (\sigma k_1, k_2, k_3, \sigma k_4)$.

If we use (2) and take the definitions of d and $\wedge$ in discrete case [1,6], then we obtain

$$F_k^{ij} = \Delta_{k_i} A_k^j - \Delta_{k_j} A_k^i + A_k^i A_{\tau_i k}^j - A_k^j A_{\tau_j k}^i, \tag{10}$$

where $\Delta_{k_i} A_k^j = A_{\tau_i k}^j - A_k^j$, $i, j = 1, 2, 3, 4$. The metric adjoint operation $*$ acts on the 2-dimensional basis elements of $K(4)$ as follows

$$\begin{aligned} &*\varepsilon_{12}^k = \varepsilon_{34}^{\tau_{12}k}, \qquad &&*\varepsilon_{13}^k = -\varepsilon_{24}^{\tau_{13}k}, \qquad &&*\varepsilon_{14}^k = \varepsilon_{23}^{\tau_{14}k}, \\ &*\varepsilon_{23}^k = \varepsilon_{14}^{\tau_{23}k}, \qquad &&*\varepsilon_{24}^k = -\varepsilon_{13}^{\tau_{24}k}, \qquad &&*\varepsilon_{34}^k = \varepsilon_{12}^{\tau_{34}k}. \end{aligned}$$

Then we obtain

$$\begin{aligned} *F = \sum_k \big(&F_{\sigma_{34}k}^{34} \varepsilon_{12}^k - F_{\sigma_{24}k}^{24} \varepsilon_{13}^k + F_{\sigma_{23}k}^{23} \varepsilon_{14}^k + \\ &+ F_{\sigma_{14}k}^{14} \varepsilon_{23}^k - F_{\sigma_{13}k}^{13} \varepsilon_{24}^k + F_{\sigma_{12}k}^{12} \varepsilon_{34}^k \big). \end{aligned} \tag{11}$$

Comparing the latter and (9) the discrete analog of the self-dual equation (the first equation of (8)) we can written as follows

$$\begin{aligned} &F_k^{12} = F_{\sigma_{34}k}^{34}, \qquad &&F_k^{13} = -F_{\sigma_{24}k}^{24}, \qquad &&F_k^{14} = F_{\sigma_{23}k}^{23}, \\ &F_k^{23} = F_{\sigma_{14}k}^{14}, \qquad &&F_k^{24} = -F_{\sigma_{13}k}^{13}, \qquad &&F_k^{34} = F_{\sigma_{12}k}^{12} \end{aligned} \tag{12}$$

for all $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$. Using (10) Equations (12) can be rewritten in the following difference form:

$$\begin{aligned} &\Delta_{k_1} A_k^2 - \Delta_{k_2} A_k^1 + A_k^1 \cdot A_{\tau_1 k}^2 - A_k^2 \cdot A_{\tau_2 k}^1 = \\ &= \Delta_{k_3} A_{\sigma_{34}k}^4 - \Delta_{k_4} A_{\sigma_{34}k}^3 + A_{\sigma_{34}k}^3 \cdot A_{\sigma_4 k}^4 - A_{\sigma_{34}k}^4 \cdot A_{\sigma_3 k}^3, \end{aligned}$$

$$\Delta_{k_1}A_k^3-\Delta_{k_3}A_k^1+A_k^1\cdot A_{\tau_1 k}^3-A_k^3\cdot A_{\tau_3 k}^1=$$
$$=-\Delta_{k_2}A_{\sigma_{24}k}^4+\Delta_{k_4}A_{\sigma_{24}k}^2-A_{\sigma_{24}k}^2\cdot A_{\sigma_4 k}^4+A_{\sigma_{24}k}^4\cdot A_{\sigma_2 k}^2,$$

$$\Delta_{k_1}A_k^4-\Delta_{k_4}A_k^1+A_k^1\cdot A_{\tau_1 k}^4-A_k^4\cdot A_{\tau_4 k}^1=$$
$$=\Delta_{k_2}A_{\sigma_{23}k}^3-\Delta_{k_3}A_{\sigma_{23}k}^2+A_{\sigma_{23}k}^2\cdot A_{\sigma_3 k}^3-A_{\sigma_{23}k}^3\cdot A_{\sigma_2 k}^2,$$

$$\Delta_{k_2}A_k^3-\Delta_{k_3}A_k^2+A_k^2\cdot A_{\tau_2 k}^3-A_k^3\cdot A_{\tau_3 k}^2=$$
$$=\Delta_{k_1}A_{\sigma_{14}k}^4-\Delta_{k_4}A_{\sigma_{14}k}^1+A_{\sigma_{14}k}^1\cdot A_{\sigma_4 k}^4-A_{\sigma_{14}k}^4\cdot A_{\sigma_1 k}^1,$$

$$\Delta_{k_2}A_k^4-\Delta_{k_4}A_k^2+A_k^2\cdot A_{\tau_2 k}^4-A_k^4\cdot A_{\tau_4 k}^2=$$
$$=-\Delta_{k_1}A_{\sigma_{13}k}^3+\Delta_{k_3}A_{\sigma_{13}k}^1-A_{\sigma_{13}k}^1\cdot A_{\sigma_3 k}^3+A_{\sigma_{13}k}^3\cdot A_{\sigma_1 k}^1,$$

$$\Delta_{k_3}A_k^4-\Delta_{k_4}A_k^3+A_k^3\cdot A_{\tau_3 k}^4-A_k^4\cdot A_{\tau_4 k}^3=$$
$$=\Delta_{k_1}A_{\sigma_{12}k}^2-\Delta_{k_2}A_{\sigma_{12}k}^1+A_{\sigma_{12}k}^1\cdot A_{\sigma_2 k}^2-A_{\sigma_{12}k}^2\cdot A_{\sigma_1 k}^1.$$

In the same way we obtain the difference anti-self-dual equation. From Equations (12) we obtain at once

$$F_k^{jr}=F_{\sigma k}^{jr} \tag{13}$$

for all $j<r,\ r=2,3,4$, where $\sigma k=(\sigma k_1,\sigma k_2,\sigma k_3,\sigma k_4)$.

Note that Equations (13) also are satisfied in the case of the difference anti-self-dual equations.

Proposition 1. *Let F be a solution of the discrete self-dual or anti-self dual equations. Then we have*

$$**F=F. \tag{14}$$

Proof. From (11) we have

$$**F=\sum_k\big(F_{\sigma_{34}k}^{34}*\varepsilon_{12}^k-F_{\sigma_{24}k}^{24}*\varepsilon_{13}^k+F_{\sigma_{23}k}^{23}*\varepsilon_{14}^k+$$
$$+F_{\sigma_{14}k}^{14}*\varepsilon_{23}^k-F_{\sigma_{13}k}^{13}*\varepsilon_{24}^k+F_{\sigma_{12}k}^{12}*\varepsilon_{34}^k\big)=$$

$$=\sum_k\big(F_{\sigma_{34}k}^{34}\varepsilon_{34}^{\tau_{12}k}+F_{\sigma_{24}k}^{24}\varepsilon_{24}^{\tau_{13}k}+F_{\sigma_{23}k}^{23}\varepsilon_{23}^{\tau_{14}k}+$$
$$+F_{\sigma_{14}k}^{14}\varepsilon_{14}^{\tau_{23}k}+F_{\sigma_{13}k}^{13}\varepsilon_{13}^{\tau_{24}k}+F_{\sigma_{12}k}^{12}\varepsilon_{12}^{\tau_{34}k}\big)=$$
$$=\sum_k\sum_{i<j}\sum_{j=2}^4F_{\sigma k}^{ij}\varepsilon_{ij}^k.$$

Comparing the latter and (13) we obtain (14).

□

It should be noted that in the case of continual Yang-Mills theory for $\mathbb{R}^4$ with the usual Euclidean metric Equation (14) is satisfied automatically for an arbitrary 2-form. But in the formalism we use the operation $(*)^2$ is equivalent to a shift.

The difference analog of Equations (13) is given by

$$\Delta_{k_j} A^r_k - \Delta_{k_r} A^j_k + A^j_k \cdot A^r_{\tau_j k} - A^r_k \cdot A^j_{\tau_r k} =$$
$$= \Delta_{k_j} A^r_{\sigma k} - \Delta_{k_r} A^j_{\sigma k} + A^j_{\sigma k} \cdot A^r_{\sigma\tau_j k} - A^r_{\sigma k} \cdot A^j_{\sigma\tau_r k},$$

where $\sigma\tau_j k = (\sigma k_1 ... k_j ... \sigma k_4)$.

3. The discrete model in Minkowski space

Let a base space of the bundle P be Minkowski space, i. e. $\mathbb{R}^4$ with the metric $g_{\mu\nu} = diag(-+++)$. In Minkowski space we write Equations (8) as

$$*F = \mp iF, \tag{15}$$

where $i^2 = -1$. Recall that F is $\mathfrak{g}$-valued, so therefore is $*F$. Then we must have $i\mathfrak{g} = \mathfrak{g}$ in obvious notation. However, this latter condition is not satisfied for the Lie algebras of any compact Lie groups G. To study Equations (15) we must choose non-compact G such as $SL(n, \mathbb{C})$ or $GL(n, \mathbb{C})$ say. This is a serious restriction since in physics the gauge groups chosen are usually compact [5]. Let the gauge group be $G = SL(2, \mathbb{C})$.

We suppose that a combinatorial model of Minkowski space has the same structure as $C(4)$. A gauge-invariant discrete model of the Yang-Mills equations in Minkowski space is given in [7]. Now the dual complex $K(4)$ is a complex of $sl(2, \mathbb{C})$-valued cochains (forms). Because discrete analogs of the differential and the exterior multiplication are not depended on a metric then they have the same form as in the case of Euclidean space. For more details on this point see [7]. However, to define a discrete analog of the $*$ operation we must take into accounts the Lorentz metric structure on $K(4)$. We denote by $\bar{x}_\kappa,\ \bar{e}_\kappa,\ \kappa \in \mathbb{Z}$ the basis elements of the 1-dimensional complex K which are corresponded to the time coordinate of Minkowski space. It is convenient to write the basis elements of $K(4) = K \otimes K \otimes K \otimes K$ in the form $\bar{\mu}^\kappa \otimes s^k$, where $\bar{\mu}^\kappa$ is either $\bar{x}^\kappa$ or $\bar{e}^\kappa$ and s^k is a basis element of $K(3) = K \otimes K \otimes K$, $k = (k_1, k_2, k_3)$, $\kappa, k_j \in \mathbb{Z}$. Then we define the $*$ operation on $K(4)$ as follows

$$\bar{\mu}^\kappa \otimes s^k \cup *(\bar{\mu}^\kappa \otimes s^k) = Q(\mu)\bar{e}^\kappa \otimes e^{k_1} \otimes e^{k_2} \otimes e^{k_3}, \tag{16}$$

where $Q(\mu)$ is equal to $+1$ if $\bar{\mu}^\kappa = \bar{x}^\kappa$ and to -1 if $\bar{\mu}^\kappa = \bar{e}^\kappa$. To arbitrary forms the $*$ operation is extended linearly. Using (16) we obtain

$$*F = \sum_k \big(F^{34}_{\sigma_{34}k}\varepsilon^k_{12} - F^{24}_{\sigma_{24}k}\varepsilon^k_{13} + F^{23}_{\sigma_{23}k}\varepsilon^k_{14} -$$
$$- F^{14}_{\sigma_{14}k}\varepsilon^k_{23} + F^{13}_{\sigma_{13}k}\varepsilon^k_{24} - F^{12}_{\sigma_{12}k}\varepsilon^k_{34} \big), \tag{17}$$

where $F^{ij}_k \in sl(2, \mathbb{C})$. Combining (17) with (9) the discrete self-dual equation $*F = iF$ can be written as follows

$$\begin{aligned} F^{34}_{\sigma_{34}k} &= iF^{12}_k, \qquad -F^{24}_{\sigma_{24}k} = iF^{13}_k, \qquad F^{23}_{\sigma_{23}k} = iF^{14}_k, \\ -F^{14}_{\sigma_{14}k} &= iF^{23}_k, \qquad F^{13}_{\sigma_{13}k} = iF^{24}_k, \qquad -F^{12}_{\sigma_{12}k} = iF^{34}_k \end{aligned} \tag{18}$$

for all $k=(k_1,k_2,k_3,k_4),\ k_r\in\mathbb{Z},\ r=1,2,3,4$. From the latter we obtain

$$F^{34}_{\sigma k}=iF^{12}_{\sigma_{12}k}=-i^2F^{34}_k=F^{34}_k,\qquad F^{24}_{\sigma k}=-iF^{13}_{\sigma_{13}k}=-i^2F^{24}_k=F^{24}_k$$

and similarly for any other components $F^{jr}_k,\ j<r$. So we have Relations (13). Thus a solution of the discrete self-dual equations (18) satisfies Equations (13) as in the Euclidean case.

We can also rewrite (18) in the difference form

$$\Delta_{k_3}A^4_{\sigma_{34}k}-\Delta_{k_4}A^3_{\sigma_{34}k}+A^3_{\sigma_{34}k}\cdot A^4_{\sigma_4 k}-A^4_{\sigma_{34}k}\cdot A^3_{\sigma_3 k}=$$
$$=i(\Delta_{k_1}A^2_k-\Delta_{k_2}A^1_k+A^1_k\cdot A^2_{\tau_1 k}-A^2_k\cdot A^1_{\tau_2 k}),$$

$$-\Delta_{k_2}A^4_{\sigma_{24}k}+\Delta_{k_4}A^2_{\sigma_{24}k}-A^2_{\sigma_{24}k}\cdot A^4_{\sigma_4 k}+A^4_{\sigma_{24}k}\cdot A^2_{\sigma_2 k}=$$
$$=i(\Delta_{k_1}A^3_k-\Delta_{k_3}A^1_k+A^1_k\cdot A^3_{\tau_1 k}-A^3_k\cdot A^1_{\tau_3 k}),$$

$$\Delta_{k_2}A^3_{\sigma_{23}k}-\Delta_{k_3}A^2_{\sigma_{23}k}+A^2_{\sigma_{23}k}\cdot A^3_{\sigma_3 k}-A^3_{\sigma_{23}k}\cdot A^2_{\sigma_2 k}=$$
$$=i(\Delta_{k_1}A^4_k-\Delta_{k_4}A^1_k+A^1_k\cdot A^4_{\tau_1 k}-A^4_k\cdot A^1_{\tau_4 k}),$$

$$-\Delta_{k_1}A^4_{\sigma_{14}k}+\Delta_{k_4}A^1_{\sigma_{14}k}-A^1_{\sigma_{14}k}\cdot A^4_{\sigma_4 k}+A^4_{\sigma_{14}k}\cdot A^1_{\sigma_1 k}=$$
$$=i(\Delta_{k_2}A^3_k-\Delta_{k_3}A^2_k+A^2_k\cdot A^3_{\tau_2 k}-A^3_k\cdot A^2_{\tau_3 k}),$$

$$\Delta_{k_1}A^3_{\sigma_{13}k}-\Delta_{k_3}A^1_{\sigma_{13}k}+A^1_{\sigma_{13}k}\cdot A^3_{\sigma_3 k}-A^3_{\sigma_{13}k}\cdot A^1_{\sigma_1 k}=$$
$$=i(\Delta_{k_2}A^4_k-\Delta_{k_4}A^2_k+A^2_k\cdot A^4_{\tau_2 k}-A^4_k\cdot A^2_{\tau_4 k}),$$

$$-\Delta_{k_1}A^2_{\sigma_{12}k}+\Delta_{k_2}A^1_{\sigma_{12}k}-A^1_{\sigma_{12}k}\cdot A^2_{\sigma_2 k}+A^2_{\sigma_{12}k}\cdot A^1_{\sigma_1 k}=$$
$$=i(\Delta_{k_3}A^4_k-\Delta_{k_4}A^3_k+A^3_k\cdot A^4_{\tau_3 k}-A^4_k\cdot A^3_{\tau_4 k}).$$

In similar manner we obtain the difference anti-self-dual equations. Obviously an anti-self-dual solution satisfies Equations (13).

Proposition 2. *Let for any $sl(2,\mathbb{C})$-valued 2-form F Conditions (13) are satisfied. Then we have*

$$**F=-F.$$

Proof. If components of any discrete 2-form F satisfy (13), then F is a solution of the discrete self-dual or anti-self-dual equations. Hence

$$**F=*(\mp iF)=\mp i*F=(\mp i)^2F=-F.$$

□

Remark. *In the continual case the self-dual and anti-self-dual equations are written in the form (15) because we have $**F=-F$ for an arbitrary 2-form F in Minkowski space. In the discrete model case it is easy to check that in $K(4)$ we have*

$$**F=-\sum_k\sum_{i<j}\sum_{j=2}^{4}F^{ij}_{\sigma k}\varepsilon^k_{ij}.$$

Thus Equations (15) are satisfied only under Conditions (13).

Theorem. *If exist some $N = (N_1, N_2, N_3, N_4)$, $N_r \in \mathbb{Z}$ such that*

$$F_k^{ij} = 0 \quad for\ any \quad |k| \geq |N|, \tag{19}$$

then Equations (15) (or (8)) have only the trivial solution $F = 0$.

Proof. Since for any solution of Equations (15) (or (8)) we have Relations (13) then the assertion is obvious.

□

Let g be a discrete 0-form

$$g = \sum_k g_k x^k,$$

where x^k is the 0-dimensional basis element of $K(4)$ and $g_k \in SU(2)$ (or $g_k \in Sl(2, \mathbb{C})$). The boundary condition (19) in terms of the connection components can be represented as: there is some discrete 0-form g such that

$$A_k^j = -(\Delta_{k_j} g_k) g_k^{-1} \quad \text{for any} \quad |k| \geq |N|.$$

It follows from Theorem 3 [6].

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